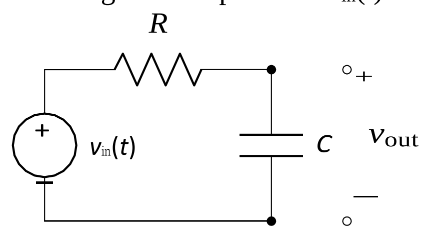
Transient response of RC and RL circuits

Resistor–capacitor (RC) and resistor–inductor (RL) circuits are the two types of *first-order* circuits: circuits either one capacitor or one inductor. In many applications, these circuits respond to a *sudden change* in an input: for example, a switch opening or closing, or a digital input switching from low to high. Just after the change, the capacitor or inductor takes some time to charge or discharge, and eventually settles on its new steady state. We call the response of a circuit immediately after a sudden change the *transient response*, in contrast to the steady state.

A first example

Consider the following circuit, whose voltage source provides $v_{in}(t) = 0$ for $t < 0$, and $v_{in}(t) = 10$ V for $t \geq 0$.

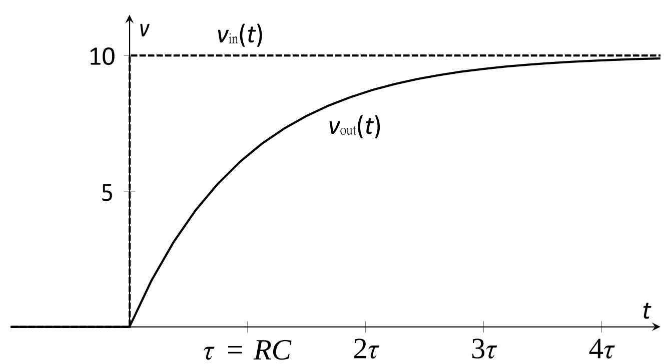


A few observations, using steady state analysis. *Just before* the step in v_{in} from 0 V to 10 V at $t = 0$, $v_{out}(0^-) = 0$ V. Since v_{out} is across a capacitor, v_{out} *just after* the step must be the same: $v_{out}(0^+) = 0$ V. *Long after* the step, if we wait long enough the circuit will reach steady state, then $v_{out}(\infty) = 10$ V. What happens in between? Using Kirchoff's current law applied at the top-right node, we could write

$$\frac{10\text{ V} - v_{out}}{R} = C\frac{dv_{out}}{dt}.$$

Solving this differential equation, then applying the initial conditions we found above, would yield (in volts)

$$v_{out}(t) = 10 - 10e^{-\frac{t}{RC}}.$$



What's happening? Immediately after the step, the current flowing through the resistor—and hence the capacitor (by KCL)—is $i(0^+) = \frac{10\text{ V}}{R}$. Since $i = C\frac{dv_{out}}{dt}$, this current causes v_{out} to start rising. This, in Turn, *reduces* the current through the resistor (and capacitor), $\frac{10\text{ V} - v_{out}}{R}$. Thus, the rate of change of v_{out} decreases as v_{out} increases. The voltage $v_{out}(t)$ technically never reaches steady state, but after about $3RC$, it's very close.

Transient response equation

It turns out that *all* first-order circuits respond to a sudden change in input with some sort of exponential decay, similar to the above. Therefore, we don't solve differential equations every time we see a capacitor or an inductor, and we won't ask you to solve any.

Instead, we use the following shortcut: In any first-order circuit, if there is a sudden change at $t = 0$, the transient response for a voltage is given by

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau},$$

where $v(\infty)$ is the (new) steady-state voltage; $v(0^+)$ is the voltage just *after* time $t = 0$; τ is the *time constant*, given by $\tau = RC$ for a capacitor or $\tau = L/R$ for an inductor, and in both cases R is the *resistance seen by the capacitor or inductor*.

The transient response for a current is the same, with $i(\cdot)$ instead of $v(\cdot)$:

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}.$$

What do we mean by the "resistance seen by the capacitor/inductor"? Informally, it means the resistance you would think the rest of the circuit had, if you were the capacitor/inductor. More precisely, you find it using these steps:

1. Zero out all sources (*i.e.* short all voltage sources, open all current sources)
2. Remove the capacitor or inductor
3. Find the resistance of the resistor network whose terminals are where the capacitor/inductor was

About the time constant

The time constant τ (the Greek letter *tau*) has units of seconds (verify, for both *RC* and *R/L*), and it governs the "speed" of the transient response. Circuits with higher τ take longer to get close to the new steady state. Circuits with short τ settle on their new steady state very quickly.

More precisely, every time constant τ, the circuit gets $1 - e^{-1} \approx 63\%$ of its way closer to its new steady state. Memorizing this fact can help you draw graphs involving exponential decays quickly.

After 3τ, the circuit will have gotten $1 - e^{-3} \approx 95\%$ of the way, and after 5τ, more than 99%. So, after a few time constants, for practical purposes, the circuit has reached steady state. Thus, the time constant is itself a good rough guide to "how long" the transient response will take.

Of course, mathematically, the steady state is actually an asymptote: it never *truly* reaches steady state. But, unlike mathematicians, engineers don't sweat over such inconsequential details.